§7.3 D'Alembert's Principle

<u>Definition 1</u> (The force of inertia):

Consider the fundamental law of motion of Newton:

$$m\vec{a} = \vec{F} \Leftrightarrow \vec{F} - m\vec{a} = 0 \quad (*)$$

We now define a vector $\vec{I}$ by

$$\vec{I} = -m\vec{a} \quad \text{"force of inertia"}$$

$\rightarrow$ equation $(*)$ becomes $F + I = 0$

$\rightarrow$ have reduced dynamics to statics

That is, by adding the force of inertia to a system, we can treat it as a static system and find its equilibrium by applying the principl of virtual work!

Adding the force of inertia, we define the effective force $\vec{F}_k^e$: $\vec{F}_k^e = \vec{F}_k + \vec{I}_k$

<u>D'Alembert's principle:</u>

The total virtual work of the effective forces is zero for all reversible variations which satisfy the given kinematical conditions:

$$\delta \overline{W^e} = \sum_{k=1}^{N} \vec{F}_k^e \cdot \delta \vec{R}_k = \sum_{k=1}^{N} (\vec{F}_k - m_k \vec{a}_k) \cdot \delta \vec{R}_k = 0$$

$$\Longleftrightarrow \quad \delta V + \underbrace{\sum m_k \vec{a}_k \cdot \delta \vec{R}_k}_{= -\delta \overline{W^i}} = 0 \qquad (**)$$

$= -\delta \overline{W^i}$ cannot be rewritten as variation of scalar function

Using $\sum m_k \ddot{\vec{R}}_k \cdot d\vec{R}_k = \sum m_k \ddot{\vec{R}}_k \cdot \dot{\vec{R}}_k \, dt$

$$= \frac{d}{dt}\Big(\underbrace{\tfrac{1}{2}\sum m_k \dot{\vec{R}}_k^2}_{=:T}\Big)dt = dT$$

we see that $(**)$ is equivalent to

$$dV + dT = d(V+T) = 0 \Rightarrow T + V = \text{const.} = E$$

"conservation of energy"

§7.4 The Lagrangian equations of motion

Hamilton's principle:

Let us multiply $\delta \overline{W^e}$ by dt and integrate between the limits $t = t_1$ and $t = t_2$:

$$\int_{t_1}^{t_2} \delta \overline{W^e} \, dt = \int_{t_1}^{t_2} \sum \Big[F_i - \frac{d}{dt}(m_i \vec{v}_i) \Big] \cdot \delta \vec{R}_i \, dt$$

The first part can be written as

$$\int_{t_1}^{t_2} \sum \vec{F}_i \cdot \delta\vec{R}_i \, dt = -\int_{t_1}^{t_2} \delta V dt = -\delta \int_{t_1}^{t_2} V \, dt$$

In the second term an integration by parts can be performed:

$$-\int_{t_1}^{t_2} \frac{d}{dt}(m_i \vec{v}_i) \cdot \delta\vec{R}_i \, dt$$

$$= \underbrace{-\int_{t_1}^{t_2} \frac{d}{dt}\left(m_i \vec{v}_i \cdot \delta\vec{R}_i\right) dt}_{= -\left[m_i \vec{v}_i \cdot \delta\vec{R}_i\right]_{t_1}^{t_2}} + \int_{t_1}^{t_2} m_i \vec{v}_i \cdot \frac{d}{dt}(\delta\vec{R}_i) dt$$

For the second term we write

$$\int_{t_1}^{t_2} m_i \vec{v}_i \cdot \frac{d}{dt} \delta\vec{R}_i \, dt = \int_{t_1}^{t_2} m_i \vec{v}_i \cdot \delta\vec{v}_i \, dt$$

$$= \frac{1}{2} \int_{t_1}^{t_2} m_i \, \delta(\vec{v}_i \cdot \vec{v}_i) \, dt = \frac{1}{2} \delta \int_{t_1}^{t_2} m_i \vec{v}_i^{\,2} \, dt$$

Summing over all particles we finally get

$$\int_{t_1}^{t_2} \delta W^e \, dt = \delta \int_{t_1}^{t_2} \frac{1}{2} \sum m_i \vec{v}_i^{\,2} dt - \delta \int_{t_1}^{t_2} V dt - \left[\sum m_i \vec{v}_i \cdot \delta\vec{R}_i\right]_{t_1}^{t_2}$$

Making use of the kinetic energy $T = \sum_i \frac{1}{2} m_i \vec{v}_i^{\,2}$ and setting $L = T - V$, finally gives:

$$\int_{t_1}^{t_2} \delta \overline{W}^e dt = \delta \int_{t_1}^{t_2} L\,dt - \left[\sum m_i \vec{v}_i \cdot \delta \vec{R}_i \right]_{t_1}^{t_2}$$

We now require that $\delta \vec{R}_i$ shall vanish at the two limits t_1 and t_2:

$$\delta R_i(t_1) = 0, \quad \delta R_i(t_2) = 0$$

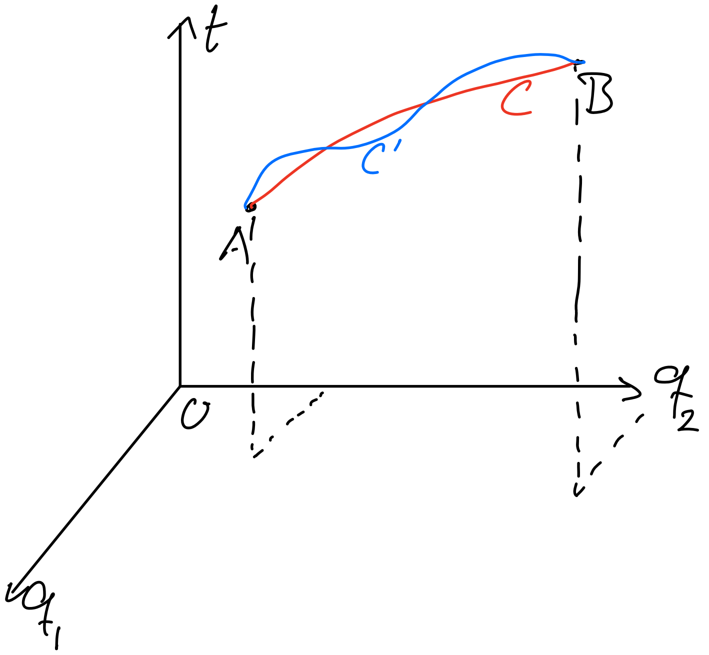
$$\rightarrow \int_{t_1}^{t_2} \delta \overline{W}^e dt = \delta \int_{t_1}^{t_2} L\,dt = \delta S,$$

where $S = \int_{t_1}^{t_2} L\,dt$

→ d'Alembert's principle can be reformulated as $\delta S = 0$ "Hamilton's principle"

Taking L to be a function of n generalized coordinates $q_1, \dots, q_n$ and n velocities $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$, we see that the solution of $\delta S = 0$ can be expressed as a curve in the $(n+1)$-dim.

configuration space of q_i and time t:

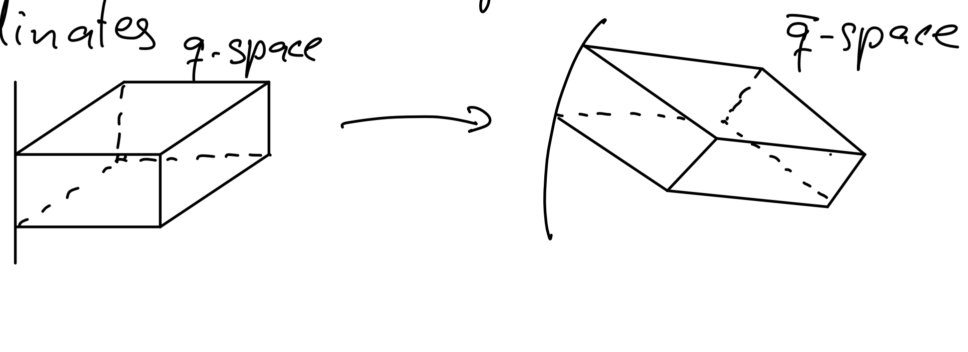


From the Euler–Lagrange eqs. we know that the necessary and sufficient conditions for $\delta S = 0$ are

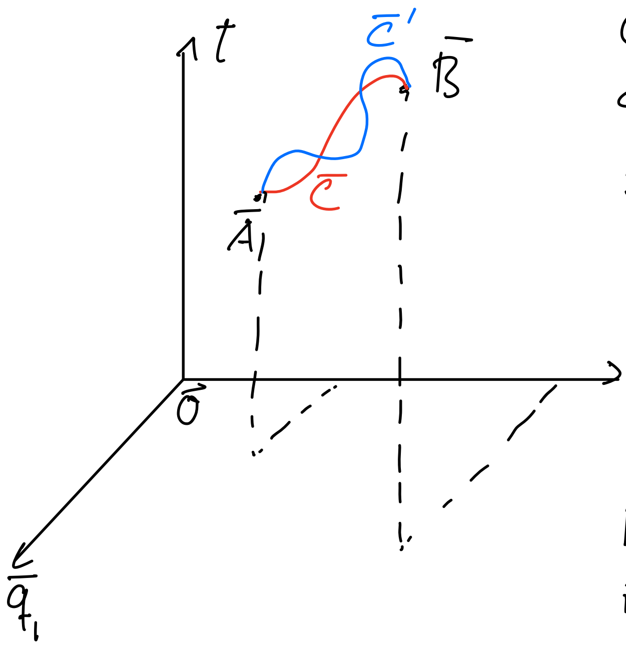
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad (i = 1, \ldots, n)$$

→ this problem in idependent from the choice of coordinates $\{q_i, t\}$!

Let us assume that the original set of coordinates is changed to a new set of coordinates



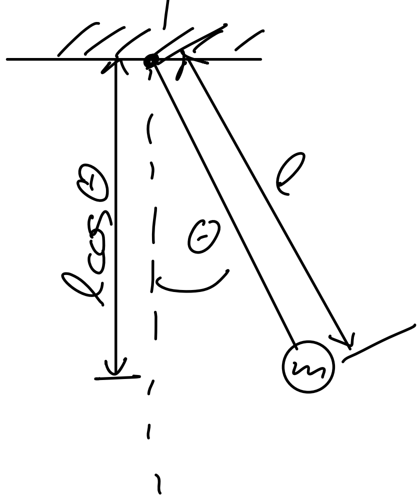
Curve C is transformed to $\bar{C}$ in $(\vec{q},t)$ - space



Curves $\bar{C}$ and $\bar{C}'$ are different from the original C and C', but the values of L remain the same along them and the form of the Euler-Lagrange eqs. is invariant!

Example 3:

Consider our familiar pendulum:



$$v = \ell \cdot \dot{\Theta} \longrightarrow T = \frac{1}{2} m v^2 = \frac{1}{2} m (\ell \cdot \dot{\Theta})^2 = \frac{1}{2} m \ell^2 \dot{\Theta}^2$$

$$y = l - l\cos\theta = l\cdot(1-\cos\theta)$$

$$\rightarrow V = mgy = mgl(1-\cos\theta)$$

$$L = T - V = \frac{1}{2}ml^2\dot\theta^2 - mgl(1-\cos\theta)$$

$\longrightarrow$ the only generalized coordinate is $q_1 = \theta$

$$\longrightarrow \frac{\partial L}{\partial \dot\theta} = ml^2\dot\theta^2$$

and $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = ml^2 \cdot \ddot\theta$$

Together with

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = ml^2\ddot\theta + mgl\sin\theta = 0$$

$$\Longleftrightarrow \quad \ddot\theta + \frac{g}{l}\sin\theta = 0$$

<u>The energy theorem</u>:

Let the virtual displacements δq_i at each instant coincide with the actual displacement dq_i which takes place during the infinitesimal time $dt = \varepsilon$

$$\rightarrow \quad \delta q_i = dq_i = \varepsilon \dot{q}_i \qquad (1)$$

This variation alters the coordinates $q_i(t)$ even at the two end points t_1 and t_2

$$\rightarrow \quad \delta \int_{t_1}^{t_2} L\, dt = \left[\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]_{t_1}^{t_2} \qquad (2)$$

Let us define "generalized momenta":

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad (3)$$

(for single free particle and cartesian coordinates p_1, p_2, p_3 become identical with rectangular components of momentum $m\vec{v}$)

Using the definition (3), the equations of motion become

$$\dot{p}_i = \frac{\partial L}{\partial q_i}$$

$\longrightarrow$ (2) becomes $\delta \int_{t_1}^{t_2} L\, dt = \left[\sum_{i=1}^{n} p_i \delta q_i \right]_{t_1}^{t_2}$

Let us assume L does not contain time explicitly, i.e.

$$L = L(q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n)$$

Then, condition (i) leads to:

$$\delta L = dL = \varepsilon \dot{L}$$

$$\rightarrow \delta \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \delta L\,dt = \int_{t_1}^{t_2} \varepsilon \dot{L}\,dt = \varepsilon \left[L\right]_{t_1}^{t_2}$$

and $\left[\sum p_i \delta q_i\right]_{t_1}^{t_2} = \varepsilon \left[\sum_{i=1}^{n} p_i \dot{q}_i\right]_{t_1}^{t_2}$

$$\rightarrow \left[\sum_{i=1}^{n} p_i \dot{q}_i - L\right]_{t_1}^{t_2} = 0$$

Since t_2 may be chosen arbitrarily, we obtain

$$\sum_{i=1}^{n} p_i \dot{q}_i - L = \text{const.}$$

Plugging in $T-V$ for L and using

$$\sum_{i=1}^{n} p_i \dot{q}_i = 2T$$

gives

$$2T - (T-V) = T + V = \text{const.} = \underline{E}$$

$\rightarrow$ conservation of energy only holds in systems where V is not explicitly t- and $\dot{q}_i$-dependent and where the kinetic energy is a quadratic form in the velocities!

§7.5 The Noether Theorem

1) Consider the case of a Lagrangian that is "translation-invariant".
In the following we use rectangular coordinates x_i, y_i, z_i and assume

$$V = V(x_i - x_k, y_i - y_k, z_i - z_k)$$

i.e the potential only depends on the "difference" of particle coordinates.

Then the transformation

$$\left.\begin{aligned} x_i &= x_i' + \alpha, \\ y_i &= y_i' + \beta, \\ z_i &= z_i' + \gamma, \end{aligned}\right\} \text{"symmetries" of Lagrangian } L$$

where α, β, γ are constants, changes neither the potential nor the kinetic energy of the system
Now let make α, β, γ t-dependent:

$$\longrightarrow T = \frac{1}{2} \sum_{i=1}^{N} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

$$= \frac{1}{2} \sum_{i=1}^{N} m_i \left[(\dot{x}_i' + \dot{\alpha})^2 + (\dot{y}_i' + \dot{\beta})^2 + (\dot{z}_i' + \dot{\gamma})^2 \right],$$

and assuming $\alpha, \beta, \gamma \ll 1$, we obtain

$$S = \int_{t_1}^{t_2} (T - V)\, dt$$

$$= \int_{t_1}^{t_2} L\, dt + \int_{t_1}^{t_2} \sum_{i=1}^{N} m_i (\dot{x}_i \dot{\alpha} + \dot{y}_i \dot{\beta} + \dot{z}_i \dot{\gamma})\, dt + \mathcal{O}(\alpha^2, \beta^2, \gamma^2)$$

→ the Lagrangian equations with respect to the new action variables α, β, γ yield

$$\left.\begin{aligned} \sum_{i=1}^{N} m_i \dot{x}_i &= c_1, \\ \sum_{i=1}^{N} m_i \dot{y}_i &= c_2, \\ \sum_{i=1}^{N} m_i \dot{z}_i &= c_3 \end{aligned}\right\} \Rightarrow \text{"conservation of momentum"}$$

2) Consider next a Lagrangian that is "rotation invariant":

V depends only on the "distance" between two particles, i.e. on

$$r_{ik} = \sqrt{(x_i - x_k)^2 + (y_i - y_k)^2 + (z_i - z_k)^2}$$

→ constant translations AND constant rotations leave both the potential and kinetic energy invariant

→ a general rotation can be written as:

$$\vec{r} = \vec{r}\,' + \vec{\Omega} \times \vec{r}\,'$$

where Ω is an arbitrary infinitesimal vector (axis of rotation).

Once again, makin Ω dependent on t, we get

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{r}_i^2 = \frac{1}{2} \sum_{i=1}^{N} m_i \left(\dot{r}_i' + \dot{\vec{\Omega}} \times \vec{r}_i\,'\right)^2$$

$$= \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{r}}_i\,'^2 + \dot{\vec{\Omega}} \sum_{i=1}^{N} m_i \left(\vec{r}_i \times \dot{\vec{r}}_i\right) + O(\Omega^2)$$

→ Euler-Lagrange equations with respect to $\vec{\Omega}$ yield: "conservation of angular momentum"

$$\sum_{i=1}^{N} m_i \left(\vec{r}_i \times \dot{\vec{r}}_i\right) = \sum_{i=1}^{N} \left(\vec{r}_i \times m_i \vec{v}_i\right) = \vec{M} = \text{const.}$$